

SPECTRAL FLOW FOR SKEW-ADJOINT FREDHOLM OPERATORS

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ABSTRACT. An analytic definition of a \mathbb{Z}_2 -valued spectral flow for paths of real skew-adjoint Fredholm operators is given. It counts the parity of the number of changes in the orientation of the eigenfunctions at eigenvalue crossings through 0 along the path. The \mathbb{Z}_2 -valued spectral flow is shown to satisfy a concatenation property and homotopy invariance, and it provides an isomorphism on the fundamental group of the real skew-adjoint Fredholm operators. Moreover, it is connected to a \mathbb{Z}_2 -index pairing for suitable paths. Applications concern the zero energy bound states at defects in a Majorana chain and a spectral flow interpretation for the \mathbb{Z}_2 -polarization in these models.

1. INTRODUCTION

The main objective of this paper is to construct a \mathbb{Z}_2 -valued spectral flow for paths of skew-adjoint Fredholms on a real Hilbert space. Our justification for using the term ‘spectral flow’ for the spectral invariant defined here is that it satisfies the three properties that can be used to axiomatize [14] the spectral flow for the self-adjoint Fredholm operators on a complex Hilbert space [2, 16], namely:

- (i) normalisation, (ii) concatenation, (iii) homotopy invariance.

In Section 2 this will first be achieved for spectral flow along straight line paths in finite dimensions. The correct definition simply counts the number of orientation changes of the eigenfunctions at eigenvalue crossings at 0 modulo 2. Hence the \mathbb{Z}_2 -spectral flow is *not* of purely spectral nature, but also depends on the eigenfunctions, as explained on a particularly simple example in Section 2. For the extension to Fredholm operators we then follow in Section 4 the analytic approach to the complex spectral flow for paths of self-adjoint Fredholm operators on a complex Hilbert space as described in [16], namely a partitioning argument is used allowing to restrict to the finite dimensional case. In the complex case, this circumvents considerable technical difficulties linked to the topologists’ intersection number approach (*e.g.* [10] based on [19]) and leads to computable formulas [7]. In the present context it allows to show relatively directly that the \mathbb{Z}_2 -valued spectral flow can be calculated, similarly to the complex spectral flow [4], as a sum of index type contributions (Section 6), provided the appropriate notion of index is used. This turns out to be the “mod 2” index map on a subgroup of the orthogonal group which was introduced in [6] and is reviewed in Section 5. Finally an index formula is proved in Section 8 which connects the \mathbb{Z}_2 -valued spectral flow of certain paths in the skew-adjoint operators on a real Hilbert space to the \mathbb{Z}_2 -index of an associated Toeplitz operator on the complexification. All of this is illustrated in Section 9 by an explicit example given by a

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matrix-valued shift operator which can be considered to be the analogue in real Hilbert space of the standard Toeplitz operator in the complex case. This example can be viewed as the canonical non-trivial example of \mathbb{Z}_2 -valued spectral flow.

Next, let us place the \mathbb{Z}_2 -valued spectral flow into the perspective of the work of Atiyah and Singer [3] on the classifying spaces for real K -theory. One of these spaces is the set $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ of skew-adjoint Fredholm operators on a separable real Hilbert space $\mathcal{H}_{\mathbb{R}}$. Its homotopy groups are known to be 8-periodic and given by

$$(1.1) \quad \begin{array}{c|cccccccc} i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \pi_i(\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})) & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & 2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} \end{array}$$

The two components of $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$, that is $\pi_0(\mathcal{F}^1(\mathcal{H}_{\mathbb{R}}))$, can be read off the parity of the kernel dimension (see [3] and Section 3 below). One of the contribution of this paper is to show that $\pi_1(\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})) \cong \mathbb{Z}_2$ is detected by the \mathbb{Z}_2 -valued spectral flow, which actually provides an explicit isomorphism (see Section 8). We do not attempt here, however, to obtain topological formulas for the \mathbb{Z}_2 -valued spectral flow using real analogues of the Atiyah-Singer index theorem. To our knowledge this is an unresolved problem. Furthermore, we do not study $\pi_1(\mathcal{F}^7(\mathcal{H}_{\mathbb{R}})) \cong \mathbb{Z}_2$ as a spectral flow here. Let us also note that several of the results below can be rewritten using Clifford valued indices as described in Section III.10 of [13], but this will not be spelled out in any detail.

While this paper focuses on the mathematical questions addressed above, our motivation comes from the use of real K -theory in mathematical physics. We present two examples of applications, both to the theory of topological insulators. The interested reader should see Section 10 for details. In this context the importance of a real spectral flow has recently been highlighted [21]. Let us point out that another notion of \mathbb{Z}_2 -spectral flow more closely linked to the complex spectral flow under a particular symmetry condition was investigated in [8, 9]. This is not connected to the one studied in the present paper.

2. \mathbb{Z}_2 -VALUED SPECTRAL FLOW FOR LINEAR PATHS IN FINITE DIMENSION

This section defines the \mathbb{Z}_2 -valued spectral flow associated to a linear (or straight line) path $t \in [0, 1] \mapsto T_t = (1 - t)T_0 + tT_1$ of skew-adjoint operators on a finite dimensional Hilbert space $\mathcal{H}_{\mathbb{R}}$. As a motivation, let us begin with $\mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$ and consider two paths, one linear and a second non-analytic path $t \in [0, 1] \mapsto \tilde{T}_t$ of skew-adjoint matrices:

$$(2.1) \quad T_t = (2t - 1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{T}_t = |2t - 1| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The spectra of T_t and \tilde{T}_t as complex operators are $\sigma(T_t) = \sigma(\tilde{T}_t) = \{(1 - 2t)\iota, (2t - 1)\iota\}$ with $\iota = \sqrt{-1}$ so that both eigenvalues form a crossing with a double degenerate kernel at $t = \frac{1}{2}$, and the associated complex spectral flow (in any possible sense, *e.g.* of [2, 16]) vanishes.

Nevertheless, there is a difference between the two paths. In fact, for \tilde{T}_t , one can consider the homotopy $s \in [0, 1] \mapsto \tilde{T}_t(s)$ of paths of skew-adjoints given by

$$\tilde{T}_t(s) = |2ts - 1| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $\tilde{T}_t(1) = \tilde{T}_t$, while $\tilde{T}_t(0)$ is a constant path with spectrum $\sigma(\tilde{T}_t(0)) = \{-i, i\}$ which is actually the straight-line path between \tilde{T}_0 and \tilde{T}_1 . Consequently the spectral crossing of the path \tilde{T}_t can be homotopically lifted. On the other hand, it is impossible to lift the kernel of T_t . This defect is encoded in the eigenfunctions as follows. Viewing T_0 and T_1 as non-degenerate skew-symmetric bilinear forms, results from linear algebra imply that there exists a real invertible matrix A such that

$$T_1 = A^* T_0 A.$$

Actually, here $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which exchanges the eigenvectors of the upper and lower branch of T_t at $t = \frac{1}{2}$. This is reflected by the sign of $\det(A)$ and this sign is by definition the \mathbb{Z}_2 -valued spectral flow $\text{Sf}_2(T_0, T_1)$ between the points T_0 and T_1 along the straight line path. This actually is true also for the complex spectral flow [4] (which is simply equal to the difference of the signatures of the end points). Let us stress again that due to the above, this \mathbb{Z}_2 -valued spectral flow is *not* only determined by the spectrum of the path, but rather depends on the eigenfunctions as well. However, we will show further below that a path having vanishing kernel throughout necessarily has a trivial \mathbb{Z}_2 -valued spectral flow. Another important difference between the two cases in (2.1) is that T_t is analytic in t , while \tilde{T}_t is not (this will be elaborated upon further down). After these words of motivation, we can go on to the formal definition of the \mathbb{Z}_2 -spectral flow.

Definition 2.1. *Suppose given two skew-adjoint operators T_0 and T_1 on a finite dimensional real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with minimal kernel dimension, namely their kernel has dimension equal to $\dim(\mathcal{H}_{\mathbb{R}}) \bmod 2$. Given an invertible A such that $T_1 = A^* T_0 A$, the \mathbb{Z}_2 -valued spectral flow (along the straight line path) from T_0 to T_1 is defined by*

$$\text{Sf}_2(T_0, T_1) = \text{sgn}(\det(A)) \in \mathbb{Z}_2.$$

In the above definition, \mathbb{Z}_2 appears as the mod 2 dimension of a vector space and also as the sign of a determinant, hence both as the additive group $\{0, 1\}$ and as the multiplicative group $\{1, -1\}$. While we freely identify these sets (namely 0 with 1, and 1 with -1), both group structures $(\mathbb{Z}_2, +)$ and (\mathbb{Z}_2, \cdot) will play a role in the following. As this is at the heart of the matter of the paper, we will be more careful about this point than may seem necessary to some readers.

Lemma 2.2. *The \mathbb{Z}_2 -valued spectral flow $\text{Sf}_2(T_0, T_1)$ is well-defined.*

Proof. It has to be checked that the definition is independent of the choice of A . Indeed, for any orthogonal O commuting with T_0 one also has $T_1 = (OA)^* T_0 (OA)$. If $\dim(\mathcal{H}_{\mathbb{R}})$ is even and T_0 is a complex structure, then O lies in the associated symplectic group and therefore

has determinant 1 so that $\text{sgn}(\det(OA)) = \text{sgn}(\det(A))$. If T_0 is not a complex structure, one can homotopically deform it to one using spectral calculus. Along this path, the sign of the determinant of A does not change. In the case of odd $\dim(\mathcal{H}_{\mathbb{R}})$, the invertibility of A assures that A sends the kernel of T_1 to the kernel of T_0 . On the remainder, the above argument in even dimension applies. \square

The definition also directly implies

Lemma 2.3. *For any invertible matrices B, C with $\det(C) > 0$, one has*

$$\text{Sf}_2(T_0, T_1) = \text{Sf}_2(T_1, T_0) = \text{Sf}_2(BT_0B^*, BT_1B^*) = \text{Sf}_2(CT_0C^*, T_1) .$$

Moreover, if T'_0 and T'_1 are skew-adjoints on another finite-dimensional real Hilbert space $\mathcal{H}'_{\mathbb{R}}$,

$$\text{Sf}_2(T_0 \oplus T'_0, T_1 \oplus T'_1) = \text{Sf}_2(T_0, T_1) + \text{Sf}_2(T'_0, T'_1) ,$$

with addition modulo 2 in $(\mathbb{Z}_2, +)$.

The following proposition indicates one use of the \mathbb{Z}_2 -valued spectral flow.

Proposition 2.4. *Let T_0 and T_1 be skew-adjoint operators on a finite dimensional Hilbert space with minimal kernel dimension. Let there exist a path $t \in [0, 1] \mapsto T_t$ from T_0 to T_1 with constant minimal kernel dimension of T_t for all t . Then $\text{Sf}_2(T_0, T_1) = 0$.*

Proof. We consider only the case of an even dimensional real Hilbert space. The path provides a continuous path $J_t = T_t|T_t|^{-1}$ of complex structures, and there thus exists a continuous path of $t \in [0, 1] \mapsto O_t$ such that $J_t = O_t^* J_0 O_t$. As $A = O_1$ is path connected to the identity, it follows that $\det(O_t) = 1$ and thus $\text{Sf}_2(T_0, T_1) = 0$. \square

Next let us discuss the so-called concatenation property of the \mathbb{Z}_2 -valued spectral flow.

Proposition 2.5. *For skew-adjoint operators T_0, T_1, T_2 having minimal kernel dimensions,*

$$(2.2) \quad \text{Sf}_2(T_0, T_2) = \text{Sf}_2(T_0, T_1) + \text{Sf}_2(T_1, T_2) ,$$

with addition modulo 2 in $(\mathbb{Z}_2, +)$.

Proof. If $T_1 = A^*T_0A$ and $T_2 = B^*T_1B$ for invertibles A, B , then $T_2 = (AB)^*T_0(AB)$. Taking determinants the claim follows. \square

Let us note that we always took care to suppose that the end points T_0 and T_1 of the path have minimal kernel dimension. Indeed, this fixes the invertible A in $T_1 = A^*T_0A$ on the kernel. If the kernels of T_0 and T_1 have different dimension, no such invertible A exists. If they are of same dimension, then such an A exists. However, the sign of $\det(A)$ depends on the choice of A (provided that the kernel dimension is larger than 1). One way out may seem to modify T_0 and T_1 by adding skew-adjoint perturbations W_0 and W_1 on the kernels such that the kernel dimension of $T_0 + W_0$ and $T_1 + W_1$ is minimal so that the above definition applies. Again, one readily checks that the outcome depends on the choices of W_0 and W_1 . In conclusion, there is no reasonable definition of the \mathbb{Z}_2 -valued spectral flow if the kernel dimension of the end points is not minimal.

On the other hand, this issue is not of importance for the concatenation of a subdivision of a path $t \in [0, 2] \mapsto T_t$ with T_0 and T_2 having minimal kernel dimension, namely if T_1 does not have minimal kernel dimension. Then one can add a skew-adjoint perturbation W_1 on the kernel of T_1 such that $T_1 + W_1$ has minimal kernel dimension. Now (2.2) holds if T_1 is replaced by $T_1 + W_1$. This is independent of the choice of W_1 because the two modifications cancel out. This fact is of considerable importance for the definition of the \mathbb{Z}_2 -valued spectral flow for arbitrary paths in infinite dimension in Section 4 below.

The final preparations in finite dimension concerns the definition of a \mathbb{Z}_2 -valued spectral flow for operators on different real Hilbert spaces. This is needed to identify suitable spectral subspaces in Section 4 below.

Proposition 2.6. *Let $\mathcal{E}, \mathcal{E}'$ and \mathcal{E}'' be three real Hilbert spaces of the same finite dimension and let T, T', T'' (resp.) be skew-adjoint operators with minimal kernel dimension on these spaces. Further let $V : \mathcal{E} \rightarrow \mathcal{E}'$, $V' : \mathcal{E}' \rightarrow \mathcal{E}''$ and $V'' : \mathcal{E}'' \rightarrow \mathcal{E}$ be three isomorphisms. If $\|V^*V - \mathbf{1}\| < 1$,*

$$\text{Sf}_2(T, V^*T'V) = \text{Sf}_2(VTV^*, T') .$$

If $\|V''((V')^)^{-1}V - \mathbf{1}\| < 1$, then, with addition modulo 2 in $(\mathbb{Z}_2, +)$,*

$$\text{Sf}_2(T, V''T''(V'')^*) = \text{Sf}_2(T, V^*T'V) + \text{Sf}_2(T', (V')^*T''V') .$$

Proof. By Lemma 2.3, $\text{Sf}_2(VTV^*, T') = \text{Sf}_2(V^*VT(V^*V)^*, V^*T'V)$. But $\|\mathbf{1} - V^*V\| < 1$ implies that $s \in [0, 1] \mapsto \mathbf{1} - s(\mathbf{1} - V^*V)$ is a path of invertibles connecting V^*V to $\mathbf{1}$. Now the last equality of Lemma 2.3 implies the first claim. Next let A, A' be invertibles such that $A^*TA = V^*T'V$ and $(A')^*T'A' = (V')^*T''V'$. Then $BTB^* = V''T''(V'')^*$ for

$$B = V''((V')^*)^{-1}(A')^*(V^*)^{-1}A^* = [V''((V')^*)^{-1}V][V^{-1}(A')^*(V^{-1})^*][A^*] .$$

Now by the same argument as above the factor in the first bracket has positive determinant by assumption. The other two factors have the same signs as $\det(A')$ and $\det(A)$. \square

Finally, we prove criteria which assure the hypothesis in Proposition 2.6.

Proposition 2.7. *Let $\mathcal{E}, \mathcal{E}'$ be subspaces of a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, possibly of infinite dimension. Let Q, Q' be the orthogonal projections on $\mathcal{E}, \mathcal{E}'$ respectively, and let $V : \mathcal{E} \rightarrow \mathcal{E}'$ be defined by $Vv = Q'v$. Suppose that for some $\epsilon < \frac{1}{2}$*

$$\|Q - Q'\| < \epsilon .$$

Then V is an isomorphism with $\|V\| \leq 1$ and $\|V^{-1}\| < 1 + 2\epsilon$. One has

$$\|V^*V - \mathbf{1}\| < 2\epsilon , \quad \|VV^* - \mathbf{1}\| < 2\epsilon .$$

Let now \mathcal{E}'' be a third subspace with orthogonal projection Q'' , and let $V' : \mathcal{E}' \rightarrow \mathcal{E}''$ and $V'' : \mathcal{E}'' \rightarrow \mathcal{E}$ be defined by $V'v' = Q''v'$ and $V''v'' = Qv''$. Suppose that, moreover,

$$\|Q' - Q''\| < \epsilon , \quad \|Q'' - Q\| < \epsilon .$$

Then V, V', V'' are isomorphisms and

$$\|V''((V')^*)^{-1}V - \mathbf{1}\| < 5\epsilon .$$

Proof. First of all, for $v = Qv \in \mathcal{E}$ and with norms in $\mathcal{H}_{\mathbb{R}}$, one has $\|Vv - v\| = \|(Q' - Q)v\| < \epsilon\|v\|$. Hence $\|V - \mathbf{1}\| < \epsilon$ and thus also $\|V^* - \mathbf{1}\| < \epsilon$. Combining $\|V^*V - \mathbf{1}\| \leq \|V^*(V - \mathbf{1})\| + \|V^* - \mathbf{1}\| < 2\epsilon$ and similarly the other way around. In particular, V is bijective. Next

$$\|Vv\|^2 = \|v\|^2 - v^*(\mathbf{1} - V^*V)v > (1 - 2\epsilon)\|v\|^2.$$

Choosing $v = V^{-1}w$ this implies $\|V^{-1}\| < (1 - 2\epsilon)^{-\frac{1}{2}} \leq 1 + 2\epsilon$. Finally

$$\begin{aligned} \|V''((V')^*)^{-1}V - \mathbf{1}\| &\leq \|(V'' - \mathbf{1})((V')^*)^{-1}V\| + \|(((V')^*)^{-1} - \mathbf{1})V\| + \|V - \mathbf{1}\| \\ &< \epsilon(1 + 2\epsilon) + 2\epsilon + \epsilon, \end{aligned}$$

which implies the last claim. \square

3. PRELIMINARIES ON SKEW-ADJOINT FREDHOLM OPERATORS

Let $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$ and $\mathcal{K}(\mathcal{H}_{\mathbb{R}})$ be the bounded and compact \mathbb{R} -linear operators on a separable real Hilbert space $\mathcal{H}_{\mathbb{R}}$. The \mathbb{C} -linear operators on its complexification $\mathcal{H}_{\mathbb{C}} = \mathbb{C} \otimes \mathcal{H}_{\mathbb{R}}$ are denoted by $\mathcal{B}(\mathcal{H}_{\mathbb{C}})$ and $\mathcal{K}(\mathcal{H}_{\mathbb{C}})$. The canonical complex conjugation \mathcal{C} on $\mathcal{H}_{\mathbb{C}}$ is given by $\mathcal{C}(\lambda\psi) = \bar{\lambda}\psi$ where $\lambda \in \mathbb{C}$ and $\psi \in \mathcal{H}_{\mathbb{R}}$. For $T \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})$ we also introduce the notations $\bar{T} = \mathcal{C}T\mathcal{C}$ and $T^t = (\bar{T})^*$ for the complex conjugate and the transpose. Note that both of these operators are \mathbb{C} -linear, even though \mathcal{C} is anti-linear. An operator $T \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})$ is called real if $\bar{T} = T$. The spectrum $\sigma(T)$ of every real operator $T \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})$ satisfies $\sigma(\bar{T}) = \sigma(T)$. An operator $T \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})$ is called skew-adjoint if $T^* = -T$. The spectrum of skew-adjoint operators lies on the imaginary axis, that is $\sigma(T) \subset i\mathbb{R}$.

Given $T \in \mathcal{B}(\mathcal{H}_{\mathbb{R}})$, an associated \mathbb{C} -linear operator also denoted by T is defined by $T(\lambda\psi) = \lambda T\psi$. This operator is real. Conversely, every real operator on $\mathcal{H}_{\mathbb{C}}$ can be restricted to $\mathcal{H}_{\mathbb{R}}$ and this restriction is clearly \mathbb{R} -linear. Thus

$$\mathcal{B}(\mathcal{H}_{\mathbb{R}}) \cong \{T \in \mathcal{B}(\mathcal{H}_{\mathbb{C}}) \mid \bar{T} = T\}, \quad \mathcal{K}(\mathcal{H}_{\mathbb{R}}) \cong \{K \in \mathcal{K}(\mathcal{H}_{\mathbb{C}}) \mid \bar{K} = K\}.$$

The spectrum $\sigma(T)$ of $T \in \mathcal{B}(\mathcal{H}_{\mathbb{R}})$ is always understood to be the spectrum of the complexification of T . In particular, the spectrum of every operator $T \in \mathcal{B}(\mathcal{H}_{\mathbb{R}})$ is invariant under complex conjugation and for a real skew-adjoint T this implies $\sigma(T) = -\sigma(T)$.

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let next $\mathcal{Q}(\mathcal{H}_{\mathbb{K}}) = \mathcal{B}(\mathcal{H}_{\mathbb{K}})/\mathcal{K}(\mathcal{H}_{\mathbb{K}})$ be the Calkin algebra and $\pi : \mathcal{B}(\mathcal{H}_{\mathbb{K}}) \rightarrow \mathcal{Q}(\mathcal{H}_{\mathbb{K}})$ the canonical projection. Then the Fredholm operators $\mathcal{F}(\mathcal{H}_{\mathbb{K}})$ are those operators $T \in \mathcal{B}(\mathcal{H}_{\mathbb{K}})$ with invertible $\pi(T) \in \mathcal{Q}(\mathcal{H}_{\mathbb{K}})$. Here the main object of study are the real skew-adjoint Fredholm operators

$$\mathcal{F}^1(\mathcal{H}_{\mathbb{R}}) = \{T \in \mathcal{F}(\mathcal{H}_{\mathbb{R}}) \mid T^* = -T\} \cong \{T \in \mathcal{B}(\mathcal{H}_{\mathbb{C}}) \mid \bar{T} = T \text{ and } T^* = -T\}.$$

The notation $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ is taken from the seminal paper [3] (although in their notation there is a supplementary \wedge). In [3] it is shown that $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ is one of the classifying spaces for real K -theory. As for any skew-adjoint operator $T \in \mathcal{B}(\mathcal{H}_{\mathbb{R}})$, the operator $iT \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})$ is self-adjoint, and spectral calculus is readily available. The essential spectrum is $\sigma_{\text{ess}}(T) = \sigma(\pi(T))$. One has

$T \in \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ if and only if $0 \notin \sigma_{\text{ess}}(T)$. Furthermore, $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ has two connected components which are distinguished by

$$(3.1) \quad \text{Ind}_1(T) = \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(T)) \bmod 2 \in \mathbb{Z}_2.$$

Again, the subindex 1 on Ind_1 is in agreement with the notations of [3]. If T is viewed as a real operator on the complexified Hilbert space, it is also given by

$$\text{Ind}_1(T) = \dim_{\mathbb{C}}(\text{Ker}_{\mathbb{C}}(T)) \bmod 2.$$

Let us briefly recall why this is a well-defined homotopy invariant. Indeed, using spectral calculus one can contract all positive and negative imaginary spectrum to one point \imath and $-\imath$, and then successively lift the degeneracy of the kernel by rank 2 perturbations, until the dimension of the kernel is either 0 or 1. Furthermore, Atiyah and Singer showed that $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ has the homotopy type as the (inductive limit) group O of orthogonal matrices, namely the homotopy groups are given by $\pi_n(\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})) = \pi_n(O)$, see (1.1). The standard example of an operator in the component with odd dimensional kernel is

$$T = \begin{pmatrix} 0 & -S \\ S^* & 0 \end{pmatrix}, \quad \text{on } \ell_{\mathbb{R}}^2(\mathbb{N}) \otimes \mathbb{R}^2,$$

where S is the unilateral right shift on $\ell_{\mathbb{R}}^2(\mathbb{N})$ with one-dimensional cokernel.

4. DEFINITION AND BASIC PROPERTIES OF THE \mathbb{Z}_2 -VALUED SPECTRAL FLOW

Let $t \in [0, 1] \mapsto T_t \in \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ be a continuous path such that the end points T_0 and T_1 have minimal kernel dimension, namely $\dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(T_0))$ and $\dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(T_1))$ are either both equal to 0 or both equal to 1. The idea in the following is to reduce the definition of the \mathbb{Z}_2 -valued spectral flow to the finite dimensional definition, essentially as for the complex spectral flow in [16]. There one splits the path into suitably chosen short pieces. We argue analogously. For $a > 0$ set

$$Q_a(t) = \chi_{(-a, a)}(\imath T_t),$$

where χ_I denotes the characteristic function on $I \subset \mathbb{R}$. This projection is real, that is $\overline{Q_a(t)} = Q_a(t)$, and is of finite dimensional range for a sufficiently small. Associated to the projections one has the restrictions $Q_a(t) T_t Q_a(t)$ which are viewed as skew-adjoint operators on $\mathcal{E}_a(t) = \text{Ran}(Q_a(t))$. These operators do not have necessarily minimal kernel dimension. This is enforced by adding a skew-adjoint perturbation R_t on the kernel of $Q_a(t) T_t Q_a(t)$. The choice of R_t is not necessarily continuous in t , as also the dimension of the kernel varies non-continuously with t . Now we introduce the following skew-adjoint operators on $\mathcal{E}_a(t)$ with minimal kernel dimension:

$$(4.1) \quad T_t^{(a)} = Q_a(t) T_t Q_a(t) + R_t.$$

By compactness, it is possible to choose a finite partition $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = 1$ of $[0, 1]$ and $a_n > 0$, $n = 1, \dots, N$, such that $t \in [t_{n-1}, t_n] \mapsto Q_{a_n}(t)$ is continuous and hence with

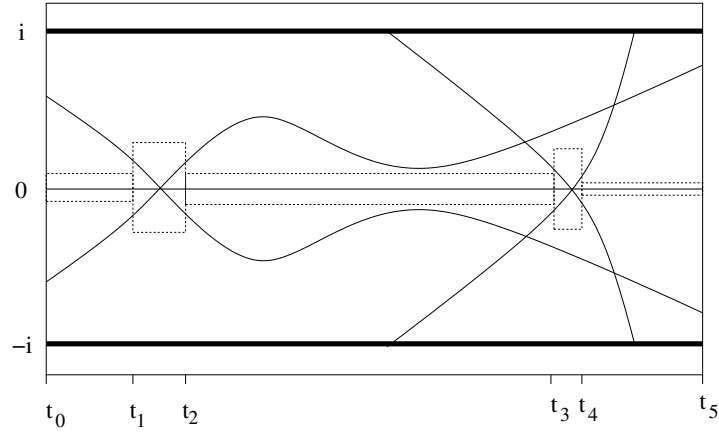


FIGURE 1. Schematic representation of the partition used for the definition of the \mathbb{Z}_2 -valued spectral flow. The vertical axis is the imaginary spectral axis of the operators T_t which have essential spectrum $\{-i, i\}$. The values a_n can be read off the heights of the dotted boxes.

constant finite rank, and, moreover, for some $\epsilon \leq \frac{1}{5}$

$$(4.2) \quad \|Q_{a_n}(t) - Q_{a_n}(t')\| < \epsilon, \quad \forall t, t' \in [t_{n-1}, t_n],$$

as well as

$$(4.3) \quad \|\pi(T_t) - \pi(T_{t'})\| < \epsilon, \quad \forall t, t' \in [t_{n-1}, t_n].$$

This is illustrated in Figure 1. Let $V_n : \mathcal{E}_{a_n}(t_{n-1}) \rightarrow \mathcal{E}_{a_n}(t_n)$ be the orthogonal projection of $\mathcal{E}_{a_n}(t_{n-1})$ onto $\mathcal{E}_{a_n}(t_n)$, namely $V_n v = Q_{a_n}(t_n)v$. By Proposition 2.6, V_n is a bijection.

Definition 4.1. For a path $t \in [0, 1] \mapsto T_t \in \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ with end points having minimal kernel dimension, let t_n and a_n as well as $T_t^{(a)}$ and V_n be as above. Then the \mathbb{Z}_2 -valued spectral flow is defined by

$$(4.4) \quad \text{Sf}_2(t \in [0, 1] \mapsto T_t) = \sum_{n=1}^N \text{Sf}_2(T_{t_{n-1}}^{(a_n)}, V_n^* T_{t_n}^{(a_n)} V_n),$$

where on the r.h.s. the Sf_2 is the finite dimensional \mathbb{Z}_2 -valued spectral flow on $\mathcal{E}_{a_n}(t_{n-1})$ defined previously, and the addition is modulo 2 in $(\mathbb{Z}_2, +)$.

The basic result on the \mathbb{Z}_2 -valued spectral flow is that it is well-defined by the above procedure.

Theorem 4.2. Let $t \in [0, 1] \mapsto T_t \in \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ be a path with end points having minimal kernel dimension. The definition of $\text{Sf}_2(t \in [0, 1] \mapsto T_t)$ is independent of the choice of the partition $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = 1$ of $[0, 1]$ and the values $a_n > 0$ such that $t \in [t_{n-1}, t_n] \mapsto Q_{a_n}(t)$ is continuous and satisfies (4.2), and also the choice of the R_t in (4.1). Moreover, Sf_2 satisfies the concatenation property with a second path $t \in [1, 2] \mapsto T_t \in \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$

$$\text{Sf}_2(t \in [0, 1] \mapsto T_t) + \text{Sf}_2(t \in [1, 2] \mapsto T_t) = \text{Sf}_2(t \in [0, 2] \mapsto T_t),$$

with addition in $(\mathbb{Z}_2, +)$, and is independent of the orientation of the path

$$\text{Sf}_2(t \in [0, 1] \mapsto T_t) = \text{Sf}_2(t \in [0, 1] \mapsto T_{1-t}) = \text{Sf}_2(t \in [0, 1] \mapsto -T_t) .$$

Proof. First let us show that adding R_t in (4.1) does not lead to an arbitrariness in the definition of Sf_2 . Indeed, suppose that $R = R_{t_n}$ is modified to R' . This changes two contributions on the r.h.s. of (4.4), namely

$$I = \text{Sf}_2(T_{t_{n-1}}^{(a_n)}, V_n^* T_{t_n}^{(a_n)} V_n) + \text{Sf}_2(T_{t_n}^{(a_{n+1})}, V_{n+1}^* T_{t_{n+1}}^{(a_{n+1})} V_{n+1}) .$$

Using Proposition 2.6 and unraveling the definitions one gets

$$\begin{aligned} I &= \text{Sf}_2(V_n T_{t_{n-1}}^{(a_n)} V_n^*, Q_{a_n}(t_n) T_{t_n} Q_{a_n}(t_n) + R) \\ &\quad + \text{Sf}_2(Q_{a_{n+1}}(t_n) T_{t_n} Q_{a_{n+1}}(t_n) + R, V_{n+1}^* T_{t_{n+1}}^{(a_{n+1})} V_{n+1}) . \end{aligned}$$

Now suppose, say, that $a_{n+1} > a_n$, and set $T_n = Q_{a_n}(t_n) T_{t_n} Q_{a_n}(t_n)$. Then there is some finite dimensional invertible real skewadjoint T'_n such that

$$Q_{a_{n+1}}(t_n) T_{t_n} Q_{a_{n+1}}(t_n) + R = T_n \oplus T'_n + R = (T_n + R) \oplus T'_n ,$$

where it was used that R only acts non-trivially on the kernel of T_n . Hence

$$I = \text{Sf}_2(V_n T_{t_{n-1}}^{(a_n)} V_n^*, T_n + R) + \text{Sf}_2((T_n + R) \oplus T'_n, V_{n+1}^* T_{t_{n+1}}^{(a_{n+1})} V_{n+1}) .$$

Now one has mod 2

$$\begin{aligned} 0 &= \text{Sf}_2(T_n + R, T_n + R') + \text{Sf}_2(T_n + R, T_n + R') + \text{Sf}_2(T'_n, T'_n) \\ &= \text{Sf}_2(T_n + R, T_n + R') + \text{Sf}_2((T_n + R) \oplus T'_n, (T_n + R') \oplus T'_n) \\ &= \text{Sf}_2(T_n + R, T_n + R') + \text{Sf}_2((T_n + R') \oplus T'_n, (T_n + R) \oplus T'_n) , \end{aligned}$$

where we appealed to Lemma 2.3. Adding this to I and using Proposition 2.5 shows

$$I = \text{Sf}_2(V_n T_{t_{n-1}}^{(a_n)} V_n^*, T_n + R') + \text{Sf}_2((T_n + R') \oplus T'_n, V_{n+1}^* T_{t_{n+1}}^{(a_{n+1})} V_{n+1}) ,$$

namely the desired independence on the choice of R . The remainder of the argument transposes [16] to the \mathbb{Z}_2 -case, notably we check that the \mathbb{Z}_2 -valued spectral flow remains unchanged under (1) refining the partition using the same a_n and (2) keeping the same partition, but changing the a_n . As to (1), let $t'_n \in (t_{n-1}, t_n)$ be added to the partition and let the associated value be $a'_n = a_n$. Then there are isomorphisms $V'_n : \mathcal{E}_{a_n}(t_{n-1}) \rightarrow \mathcal{E}_{a_n}(t'_n)$ and $V''_n : \mathcal{E}_{a_n}(t'_{n-1}) \rightarrow \mathcal{E}_{a_n}(t_n)$ defined via orthogonal projections as above. Then we claim that

$$\text{Sf}_2(T_{t_{n-1}}^{(a_n)}, V_n^* T_{t_n}^{(a_n)} V_n) = \text{Sf}_2(T_{t_{n-1}}^{(a_n)}, (V'_n)^* T_{t'_n}^{(a_n)} V'_n) + \text{Sf}_2(T_{t'_n}^{(a_n)}, (V''_n)^* T_{t_n}^{(a_n)} V''_n) ,$$

with addition in \mathbb{Z}_2 . This actually follows from Propositions 2.6 and 2.7. For (2), let us suppose that there are $a'_n > a_n$ both satisfying the (4.2). In particular, both $Q_{a_n}(t)$ and $Q_{a'_n}(t)$ have constant dimension, and thus also the projection $Q_{a'_n}(t) - Q_{a_n}(t)$ has constant dimension. This implies that the added eigenvalues remain in (a_n, a'_n) and $(-a'_n, a_n)$ and thus do not contribute

to the \mathbb{Z}_2 -valued spectral flow by combining the additivity of Lemma 2.3. Finally, according to Lemma 2.3,

$$\text{Sf}_2(T_{t_{n-1}}^{(a_n)}, V_n^* T_{t_n}^{(a_n)} V_n) = \text{Sf}_2(V_n T_{t_{n-1}}^{(a_n)} V_n^*, T_{t_n}^{(a_n)}) = \text{Sf}_2(T_{t_n}^{(a_n)}, V_n T_{t_{n-1}}^{(a_n)} V_n^*).$$

This implies the last claim. \square

The proof of the following result is exactly as that of Proposition 3 in [16] and Proposition 2.5 in [17], provided the local concatenation of spectral flow (based on Lemma 1.3 in [17]) is replaced by Proposition 2.6.

Theorem 4.3. *Let $t \in [0, 1] \mapsto T_t$ and $t \in [0, 1] \mapsto T'_t$ be two continuous paths in $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ such that $T_0 = T'_0$ and $T_1 = T'_1$ have minimal kernel dimension. If the two paths are connected via a continuous homotopy leaving the endpoints fixed, $\text{Sf}_2(t \in [0, 1] \mapsto T_t) = \text{Sf}_2(t \in [0, 1] \mapsto T'_t)$.*

At this point, one may be tempted to write simply $\text{Sf}_2(T_0, T_1)$ for $\text{Sf}_2(t \in [0, 1] \mapsto T_t)$. This is, however, not possible because the \mathbb{Z}_2 -valued spectral flow truly depends on the path, and not only on the end points. Indeed, there exist non-trivial loops based at T_1 which when concatenated with $t \in [0, 1] \mapsto T_t$ change the value of the \mathbb{Z}_2 -valued spectral flow. However, in the case where $\mathcal{H}_{\mathbb{R}}$ is finite dimensional and $t \in [0, 1] \mapsto T_t$ is the linear path (or a homotopy of it), one has $\text{Sf}_2(t \in [0, 1] \mapsto T_t) = \text{Sf}_2(T_0, T_1)$.

As a final issue, let us consider (real) analytic paths in $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$. By analytic perturbation theory [12, VII.3], all eigenvalues and eigenvectors can be chosen to be analytic. In particular, T_t has minimal kernel dimension except on a discrete set of crossings. At each of these crossings, one has blocks as in the first example of (2.1), and not the second. Hence each crossing (of simple multiplicity) contributes a unit to the \mathbb{Z}_2 -valued spectral flow, and we can conclude the following:

Theorem 4.4. *Let $t \in [0, 1] \mapsto T_t \in \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ be analytic and have end points with minimal kernel dimension. Then $\text{Sf}_2(t \in [0, 1] \mapsto T_t)$ is modulo 2 equal to the sum of all eigenvalue crossings through 0 along the path, each one counted with its multiplicity.*

5. INDEX MAP ON THE ORTHOGONAL GROUP

The aim of the following two sections is to give an alternative description of the \mathbb{Z}_2 -valued spectral flow, that is to express it in terms of the index map on the orthogonal group introduced in [6]. This section reviews the construction and main properties of this index map from [6], but is kept self-contained with all proofs given in a slightly generalized form which will be used in Section 6 to establish the alternative description. Recall that the orthogonal group on a real Hilbert space is defined as

$$\mathcal{O}(\mathcal{H}_{\mathbb{R}}) = \{O \in \mathcal{B}(\mathcal{H}_{\mathbb{R}}) \mid O^*O = \mathbf{1}\}.$$

If the orthogonal operators are viewed as operators on the complexification $\mathcal{H}_{\mathbb{C}}$, then $\mathcal{O}(\mathcal{H}_{\mathbb{R}})$ can be identified with the real unitaries:

$$\mathcal{O}(\mathcal{H}_{\mathbb{R}}) \cong \{O \in \mathcal{B}(\mathcal{H}_{\mathbb{C}}) \mid O^*O = \mathbf{1} \text{ and } \overline{O} = O\}.$$

Let us first suppose that $\dim_{\mathbb{R}}(\mathcal{H}_{\mathbb{R}}) < \infty$. Then one of the basic facts is that $\mathcal{O}(\mathcal{H}_{\mathbb{R}})$ has two connected components which can be distinguished by the map $j : (\mathcal{O}(\mathcal{H}_{\mathbb{R}}), \cdot) \rightarrow (\mathbb{Z}_2, \cdot)$ defined by

$$j(O) = \text{sgn}(\det(O)) ,$$

where $\det(\mathbf{1}) = 1$. This is clearly a homomorphism. Let us suppose that $\dim_{\mathbb{R}}(\mathcal{H}_{\mathbb{R}})$ is even and J is a given fixed complex structure on $\mathcal{H}_{\mathbb{R}}$. The one can rewrite j as

$$(5.1) \quad j(O) = \frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(J + OJO^*)) \bmod 2$$

$$(5.2) \quad = \frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(O - JOJ)) \bmod 2$$

$$(5.3) \quad = \frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(\mathbf{1} - \frac{1}{2}O^*J[O, J])) \bmod 2 ,$$

where in (5.1) the two representations of \mathbb{Z}_2 are identified, as described after Definition 2.1. To verify (5.1), let us note that $\text{sgn}(\det(O))$ is homotopy invariant and so is the r.h.s. of (5.1) (see [6] or Proposition 5.2 below), and the equality can readily be checked to hold for two points in the two components. Let us also point out that (5.1) is independent of the choice of J . The other equalities (5.2) and (5.3) then readily follow. If the dimension of $\mathcal{H}_{\mathbb{R}}$ is odd, then there is no complex structure and (5.1) does not hold.

In infinite dimension, it is known from Kuipers' theorem that $\mathcal{O}(\mathcal{H}_{\mathbb{R}})$ is contractible. On the other hand, (5.3) suggests that an invariant can be defined whenever the commutator $[O, J]$ is compact for a given fixed complex structure J (which always exist on an infinite dimensional Hilbert space). Hence let us set, as in [6],

$$(5.4) \quad \mathcal{O}_J(\mathcal{H}_{\mathbb{R}}) = \{O \in \mathcal{O}(\mathcal{H}_{\mathbb{R}}) \mid [O, J] \in \mathcal{K}(\mathcal{H}_{\mathbb{R}})\} .$$

This is actually a subgroup which, as we shall see shortly, is not connected any more. As the orthogonal group acts transitively on the set of complex structures, the subgroups associated to different J are isomorphic. More precisely, if $J' = WJW^*$ for some $W \in \mathcal{O}(\mathcal{H}_{\mathbb{R}})$, one has $\mathcal{O}_{J'}(\mathcal{H}_{\mathbb{R}}) = W\mathcal{O}_J(\mathcal{H}_{\mathbb{R}})W^*$.

Theorem 5.1. [6] *The map $j : (\mathcal{O}_J(\mathcal{H}_{\mathbb{R}}), \cdot) \rightarrow (\mathbb{Z}_2, \cdot)$ is a homotopy invariant homomorphism and labels the two connected components of $\mathcal{O}_J(\mathcal{H}_{\mathbb{R}})$.*

Furthermore, it is proved in [6] that $\mathcal{O}_J(\mathcal{H}_{\mathbb{R}})$ is of the same homotopy type as the loop space of $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ (this is also shown in the proof of Theorem 7.1 below). Theorem 5.1 follows from the following more general continuity result upon setting $J_1 = J$ and $J_2 = OJO^*$.

Proposition 5.2. *Introduce the set*

$$\mathcal{P} = \left\{ (J_0, J_1) \in \mathcal{B}(\mathcal{H}_{\mathbb{R}}) \times \mathcal{B}(\mathcal{H}_{\mathbb{R}}) \mid J_i^2 = -\mathbf{1} \text{ and } J_i^* = -J_i \text{ for } i = 0, 1, \ ||J_1 - J_0||_{\mathcal{Q}} < \frac{1}{2} \right\} ,$$

equipped with the norm topology. Then

$$(5.5) \quad (J_0, J_1) \in \mathcal{P} \mapsto \left(\frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(J_0 + J_1)) \right) \bmod 2 \in \mathbb{Z}_2 ,$$

is continuous.

The proof rests on the following lemma.

Lemma 5.3. *With the assumptions of Proposition 5.2 on the pair $J_0, J_1 \in \mathcal{P}$, let*

$$T_0 = \frac{1}{2}(J_0 + J_1), \quad T_1 = \frac{1}{2}(J_0 - J_1).$$

Then the following identities hold:

$$T_0^*T_0 + T_1^*T_1 = \mathbf{1} = T_0T_0^* + T_1T_1^*, \quad T_0^*T_1 + T_1^*T_0 = 0 = T_0T_1^* + T_1T_0^*,$$

as well as

$$T_0J_0 = J_1T_0, \quad T_0J_1 = J_0T_0, \quad T_1J_0 = -J_1T_1, \quad T_1J_1 = -J_0T_1.$$

Proof. The arguments are purely algebraic. Let us start from

$$T_0^*T_0 = -T_0^2 = -\frac{1}{4}(J_0^2 + J_1^2 + J_0J_1 + J_1J_0) = \frac{1}{2} - \frac{1}{4}(J_0J_1 + J_1J_0),$$

as well as

$$T_1^*T_1 = -T_1^2 = \frac{1}{2} + \frac{1}{4}(J_1J_0 + J_0J_1).$$

The first identity now follows and second one uses the same algebraic relations. The final group of identities are all proved in the same way, for example: $T_0J_0 = \frac{1}{2}(-\mathbf{1} + J_1J_0)$ while $J_1T_0 = \frac{1}{2}(J_1J_0 - \mathbf{1})$ which gives the first identity. \square

Proof of Proposition 5.2. By the assumption on the norm of the difference $J_1 - J_0$ in the Calkin algebra, T_0 is a skew-adjoint Fredholm. The focus is on $\text{Ker}(T_0) = \text{Ker}(T_0^*T_0)$. First of all, the identities in Lemma 5.3 imply

$$T_0^*T_0J_0 = -T_0(T_0J_0) = -T_0(J_1T_0) = -(T_0J_1)T_0 = -(J_0T_0)T_0 = J_0(T_0^*T_0).$$

Hence $T_0^*T_0$ is a complex linear operator on $\mathcal{H}_{\mathbb{R}}$ equipped with J_0 as a complex structure. This implies that $\text{Ker}(T_0) = \text{Ker}(T_0^*T_0)$ is always even dimensional as a real vector space. It follows that the map in (5.5) is well-defined and really takes values in $\mathbb{Z}_2 = \{0, 1\}$.

We now claim that all eigenvalues $\lambda \in (0, 1)$ of the non-negative Fredholm operator $T_0^*T_0$ have even complex multiplicity, which implies that their real multiplicity is divisible by 4. To see this, let a non-vanishing $v \in \mathcal{H}_{\mathbb{R}}$ be such that $T_0^*T_0v = \lambda v$ with $\lambda \in (0, 1)$. Then J_0v , its multiple by the imaginary unit, is also an eigenvector of $T_0^*T_0$ with eigenvalue λ . It is linearly independent of v over the reals, but in the complex Hilbert space it is, of course, linearly dependent on v . Moreover, the relations of Lemma 5.3 show that $w = T_1^*T_0v$ is also an eigenvector of $T_0^*T_0$:

$$T_0^*T_0w = T_0^*T_0T_1^*T_0v = -T_0^*T_1T_0^*T_0v = T_1^*T_0T_0^*T_0v = T_1^*T_0\lambda v = \lambda w.$$

The norm of this vector is given by $\|w\|^2 = v^*T_0^*(1 - T_0T_0^*)T_0v = \lambda(1 - \lambda)\|v\|^2$ so that it is non-vanishing for $\lambda \neq 0, 1$, and furthermore w is linearly independent of v as complex vector. Indeed, suppose that $w = (\mu_0 + \mu_1J_0)v$ with some $\mu_0, \mu_1 \in \mathbb{R}$ not both zero. Then applying $T_0^*T_1$ yields, again using the relations of Lemma 5.3,

$$\begin{aligned} T_0^*T_1T_1^*T_0v = T_0^*T_1(\mu_0 + \mu_1J_0)v &\implies T_0^*T_0(\mathbf{1} - T_0^*T_0)v = -(\mu_0 - \mu_1J_0)T_1^*T_0v \\ &\implies \lambda(1 - \lambda)v = -(\mu_0 - \mu_1J_0)w \\ &\implies \lambda(1 - \lambda)(\mu_0 + \mu_1J_0)v = -(\mu_0^2 + \mu_1^2)w, \end{aligned}$$

where in the last implication we applied $\mu_0 + \mu_1 J_0$. Hence $\lambda(1 - \lambda)w = -(\mu_0^2 + \mu_1^2)w$ which is a contradiction because $\lambda \leq 1$. Thus v, w span a two dimensional complex Hilbert space of eigenvectors for $T_0^* T_0$ with eigenvalue λ .

Suppose that u is another eigenvector of $T_0^* T_0$ with eigenvalue λ that is orthogonal to the real span \mathcal{E} of $\{v, J_0 v, T_1^* T_0 v, J_0 T_1^* T_0 v\}$. Then the span of $\{u, J_0 u, T_1^* T_0 u, J_0 T_1^* T_0 u\}$ can be seen to be orthogonal to \mathcal{E} showing that each eigenspace of $T_0^* T_0$ is a direct sum of these four (real) dimensional subspaces. Given that the degeneracy of every positive eigenvalue of $T_0^* T_0$ is divisible by 4, the result now follows. (Note that this argument is partly modeled on that in Proposition 5.1 of [3].) \square

Remark Let us stress that the above also proves the following somewhat surprising fact from linear algebra. For two complex structures J_0 and J_1 on \mathbb{R}^{2n} , the multiplicity of every eigenvalue of $(J_0 + J_1)^2$ in $(-1, 0)$ is divisible by 4. \diamond

In the remainder of this section, we provide an alternative formula for the map j .

Proposition 5.4. *Every $O \in \mathcal{O}_J(\mathcal{H}_{\mathbb{R}})$ can be written as $O = U(1 + K)$ with an orthogonal $U \in \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ satisfying $JU = UJ$ and a compact operator $K \in \mathcal{K}(\mathcal{H}_{\mathbb{R}})$. One has*

$$(5.6) \quad j(O) = \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(K + 2\mathbf{1})) \bmod 2.$$

Moreover, $\mathcal{O}_J(\mathcal{H}_{\mathbb{R}})$ can be retracted to the subgroup $\mathcal{O}_{\mathcal{K}}(\mathcal{H}_{\mathbb{R}}) = \{O \in \mathcal{O}(\mathcal{H}_{\mathbb{R}}) \mid O - \mathbf{1} \in \mathcal{K}(\mathcal{H}_{\mathbb{R}})\}$.

Proof. The first claim is Proposition 2.1 in [6], but we here provide an explicit formula for U . Set $S_0 = \frac{1}{2}(O - JOJ)$ and $S_1 = \frac{1}{2}(O + JOJ)$ so that $O = S_0 + S_1$. The formulas from Lemma 5.3 will be used for $J_1 = J$ and $J_2 = O^* JO$. As then $S_0 = OT_0 J^*$ and $S_1 = OT_1 J^*$, one has

$$S_0^* S_0 + S_1^* S_1 = \mathbf{1} = S_0 S_0^* + S_1 S_1^*, \quad S_0 J = J S_0, \quad S_1 J = -J S_1.$$

Let us use the polar decomposition $S_0 = V|S_0|$. Then $VJ = JV$, but, in general, V is only a partial isometry with kernel $\text{Ker}_{\mathbb{R}}(S_0) = \text{Ker}_{\mathbb{R}}(|S_0|)$. This kernel is J -invariant and real, and therefore even dimensional. Let us choose an operator I on $\text{Ker}_{\mathbb{R}}(S_0)$ with $I^2 = \mathbf{1}$ and $IJ = -JI$. Then the multiplicities of 1 and -1 as eigenvalues of I are equal. Now define U as V on $\text{Ker}_{\mathbb{R}}(S_0)^{\perp}$ and $S_1 I$ on $\text{Ker}_{\mathbb{R}}(S_0)$. Then U is orthogonal and satisfies $UJ = JU$. Furthermore, $O = U(|S_0| + U^* S_1)$ so that $K = |S_0| - \mathbf{1} + U^* S_1$, which is indeed compact because $S_1 = \frac{1}{2}J[O, J]$ is compact. Moreover, $U^* S_1 = I$ on $\text{Ker}_{\mathbb{R}}(S_0)$. Thus the multiplicity of -1 as eigenvalue of $\mathbf{1} + K$ is equal to $\frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(S_0))$. Comparing with (5.1), this shows the formula for j . The final claim follows directly from Kuipers' theorem because U is unitary on $\mathcal{H}_{\mathbb{R}}$ viewed as complex Hilbert space with imaginary unit J . \square

Example Given a one-dimensional projection P on $\mathcal{H}_{\mathbb{R}}$, let us set $O = \mathbf{1} - 2P$. Then $O \in \mathcal{O}_{\mathcal{K}}(\mathcal{H}_{\mathbb{R}})$ and $j(O) = 1$ by (5.6). If, moreover, $PJP = 0$ holds, one can readily check the identity $\mathbf{1} - \frac{1}{2}O^* J[O, J] = \mathbf{1} - 2P$ so that also (5.3) leads to $j(O) = 1$. \diamond

Remark An alternative proof of (5.6) can be given as follows. The r.h.s. of (5.6) is a homotopy invariant because the spectrum of every orthogonal $\mathbf{1} + K$ is invariant under complex conjugation

so that the parity of the -1 eigenvalue is conserved. By Theorem 5.1 also j is a homotopy invariant. Hence it is sufficient to check the equality on the two components. This is trivial for the identity component and was verified on the other component in the example above. \diamond

6. ALTERNATIVE FORMULATION OF THE \mathbb{Z}_2 -VALUED SPECTRAL FLOW

Let us begin by considering the straight-line path connecting two complex structure on a real Hilbert space. The following lemma on the spectral properties along this path is elementary.

Lemma 6.1. *Let J_0 and J_1 be complex structures on $\mathcal{H}_{\mathbb{R}}$ such that $\|\pi(J_0) - \pi(J_1)\|_{\mathcal{Q}} \leq c < 1$. Set $T_t = (1 - t)J_0 + tJ_1$ for $t \in [0, 1]$. Then $t \in [0, 1] \mapsto T_t$ is a path in $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ such that $\sigma_{\text{ess}}(T_t) \cap [-\iota(1 - ct), \iota(1 - ct)] = \emptyset$. Furthermore, $0 \in \sigma(T_t)$ implies that $t = \frac{1}{2}$.*

Given the situation of Lemma 6.1, $t \in [0, 1] \mapsto \iota T_t$ is a path of self-adjoint Fredholms and it is hence possible to consider the associated spectral flow $\text{Sf}(t \in [0, 1] \mapsto \iota T_t)$ in the sense of [17]. Note that as this path is analytic in t , analytic perturbation theory for the discrete spectrum of the self-adjoints ιT_t on $\mathcal{H}_{\mathbb{C}}$ applies so that all notions of spectral flow coincide. In particular, all eigenvalues crossings through 0 at $t = \frac{1}{2}$ result from analytic curves. Hence the spectral symmetry $\sigma(\iota T_t) = -\sigma(\iota T_t)$ implies that each analytic curve of an eigenvalue has a reflected partner, and the total spectral flow resulting from each such a pair vanishes. In conclusion, $\text{Sf}(t \in [0, 1] \mapsto \iota T_t) = 0$. Now the \mathbb{Z}_2 -valued spectral flow counts the number of eigenvalue exchanges at $t = \frac{1}{2}$. An important point is that the present path is analytic, hence locally at the crossing the first example in (2.1) is a good model, while the second is *not*. Therefore the number of crossings and thus the \mathbb{Z}_2 -valued spectral flow is given by the r.h.s. of (6.1). This will be shown in more detail below. As is already hinted at in Lemma 6.1, the number of these crossings can be read off from the kernel dimension of $T_{\frac{1}{2}} = \frac{1}{2}(J_0 + J_1)$. Before stating this result, let us introduce the following

Notation: $\text{Sf}_2(J_0, J_1)$ denotes $\text{Sf}_2(t \in [0, 1] \mapsto T_t)$ for the linear path described in Lemma 6.1.

Proposition 6.2. *Let J_0 and J_1 be complex structures on $\mathcal{H}_{\mathbb{R}}$ such that $\|\pi(J_0) - \pi(J_1)\|_{\mathcal{Q}} \leq c < 1$. Then the \mathbb{Z}_2 -valued spectral flow from J_0 to J_1 is*

$$(6.1) \quad \text{Sf}_2(J_0, J_1) = \left(\frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(J_0 + J_1)) \right) \bmod 2.$$

Note that as $J_0 + J_1$ is in the component of $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ with vanishing \mathbb{Z}_2 -index, the kernel of $J_0 + J_1$ is always even dimensional (as real vector space if $J_0 + J_1$ is viewed as an \mathbb{R} -linear operator, and as a complex vector space if $J_0 + J_1$ is viewed as an \mathbb{C} -linear operator). Hence the r.h.s. of (6.1) is indeed well-defined in \mathbb{Z}_2 .

Proof of Proposition 6.2. Let us calculate $\text{Sf}_2(J_0, J_1)$ as given by definition (4.4) for the special straight line path of Lemma 6.1. It is possible to choose a splitting $t_0 = 0 < t_1 < \frac{1}{2} < t_2 = 1 - t_1 < t_3 = 1$ of $[0, 1]$ in three intervals as well as $a > b$ with the following properties (see one of the crossings in Figure 1): $Q_b(t) = 0$ for $t \in [t_0, t_1]$ and $Q_b(t) = 0$ for $t \in [t_2, t_3]$, and $\text{Tr}(Q_a(t)) = \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(T_{\frac{1}{2}}))$ for $t \in [t_1, t_2]$. Note that, in particular, $\sigma(T_{\frac{1}{2}}^{(a)}) = \{0\}$. Only

the interval $[t_1, t_2] = [t_1, 1 - t_1]$ contributes to the \mathbb{Z}_2 -valued spectral flow. Hence, with the notations of Section 4,

$$\text{Sf}_2(J_0, J_1) = \text{Sf}_2(T_{t_1}^{(a)}, V_2^* T_{t_2}^{(a)} V_2) .$$

Now both finite dimensional skew-adjoint operators $T_{t_1}^{(a)}$ and $V_2^* T_{t_2}^{(a)} V_2$ are non-degenerate. Using the polar decomposition, each operator $T_t^{(a)}$ with $t \neq \frac{1}{2}$ can hence be homotopically deformed to complex structures $J_t^{(a)}$. Actually, if J_t is the (skew-adjoint) phase of T_t for $t \neq \frac{1}{2}$, then $J_t^{(a)} = Q_a(t) J_t Q_a(t)$. As there is no kernel along both of these homotopies,

$$\text{Sf}_2(J_0, J_1) = \text{Sf}_2(J_{t_1}^{(a)}, V_2^* J_{t_2}^{(a)} V_2) .$$

As the orthogonal group acts transitively on complex structures, there exists an orthogonal O such that $J_{t_2}^{(a)} = O^* J_{t_1}^{(a)} O$. From the definition of Sf_2 and (5.1), one now has

$$\text{Sf}_2(J_0, J_1) = \text{sgn}(\det(O)) = \frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(J_{t_1}^{(a)} + V_2^* J_{t_2}^{(a)} V_2)) \bmod 2 .$$

On the other hand, J_0 is homotopic to J_{t_1} and J_1 is homotopic to J_{t_2} . Thus by Proposition 5.2

$$\frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(J_0 + J_1)) \bmod 2 = \frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(J_{t_1} + J_{t_2})) \bmod 2 .$$

Moreover, T_{t_1} and T_{t_2} can be made arbitrarily close by sending t_1 to $\frac{1}{2}$ (but *not* J_{t_1} and J_{t_2}). Due to the continuity of the associated Riesz projections, V_2 can hence be extended to an invertible operator on all $\mathcal{H}_{\mathbb{R}}$ which is close to the identity and satisfies $\mathbf{1} - J_{t_1}^{(a)} = V_2^*(\mathbf{1} - J_{t_2}^{(a)})V_2$. Again using the homotopy invariance of Proposition 5.2 to deform V_2 to the identity, one concludes

$$\begin{aligned} \frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(J_0 + J_1)) \bmod 2 &= \frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(J_{t_1} + V_2^* J_{t_2} V_2)) \bmod 2 \\ &= \frac{1}{2} \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(J_{t_1}^{(a)} + V_2 J_{t_2}^{(a)} V_2^*)) \bmod 2 , \end{aligned}$$

where in the second equality we used $V_2^* J_{t_2} V_2 = \mathbf{1} - J_{t_1}^{(a)} + V_2^* J_{t_2}^{(a)} V_2$. Combined with the above, this concludes the proof. \square

Now we can write out the alternative formulation of the \mathbb{Z}_2 -valued spectral flow defined in (4.4). For the sake of simplicity, let us restrict to a norm continuous path $t \in [0, 1] \mapsto T_t \in \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ in the space of skew-adjoint real Fredholm operators with even-dimensional kernel, namely $\text{Ind}_1(T_t) = 0$. The end points T_0 and T_1 are supposed to have trivial kernel. Associated to T_t is the phase $W_t = T_t |T_t|^{-1}$. One has $(W_t)^* = -W_t$, and $(W_t)^* W_t = -W_t^2$ is the projection onto the orthogonal complement of the kernel of T_t . Furthermore, $\pi(W_t) \in \mathcal{Q}(\mathcal{H}_{\mathbb{R}})$ is a complex structure in the Calkin algebra, namely $\pi(W_t)$ is skew-adjoint and squares to minus the identity. The map $t \mapsto \pi(W_t)$ is continuous, but $t \mapsto W_t$ is *not*, as it is discontinuous at points where the kernel dimension of T_t changes. In any case, one can complete W_t on its kernel by an arbitrary complex structure to obtain a complex structure J_t on $\mathcal{H}_{\mathbb{R}}$, similar as in (4.1). Now let $t_0 = 0 < t_1 < \dots < t_n = 1$ be a partition as in (4.4). Let us further assume the partition to be sufficiently fine so that each pair $J_{t_{n-1}}, J_{t_n}$ satisfies the assumptions of Proposition 6.2 (this is possible because the difference $\pi(J_{t_{n-1}} - J_{t_n})$ can be written out using Riesz projections

and resolvent identity and then estimated using (4.3)). Then the concatenation property of Sf_2 implies that, with addition modulo 2 in $(\mathbb{Z}_2, +)$,

$$\text{Sf}_2(t \in [0, 1] \mapsto T_t) = \sum_{n=1}^N \text{Sf}_2(J_{t_{n-1}}, J_{t_n}) .$$

Let us now assume the partition to be sufficiently fine so that each pair $J_{t_{n-1}}, J_{t_n}$ satisfies the assumptions of Proposition 6.2. Then we obtain the following formula, which is the \mathbb{Z}_2 -equivalent of the index formulation of the complex spectral flow given in [4, 8, 17].

Proposition 6.3. *Let $t \in [0, 1] \mapsto T_t \in \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ be a continuous path with end points having trivial kernel. Let J_t be complex structures obtained by completing the phase $T_t|T_t|^{-1}$ by an arbitrary complex structure on the kernel. Then, for a sufficiently fine partition t_n satisfying $\|\pi(J_n - J_{n-1})\| < 1$, one has*

$$\text{Sf}_2(t \in [0, 1] \mapsto T_t) = \left(\frac{1}{2} \sum_{n=1}^N \dim_{\mathbb{R}}(\text{Ker}_{\mathbb{R}}(J_{t_{n-1}} + J_{t_n})) \right) \bmod 2 .$$

7. THE ISOMORPHISM ON THE FUNDAMENTAL GROUP

Let us first note that for loops there is no need to impose any conditions on the kernel dimension of the end point. Thus Sf_2 is a well-defined map on the set of loops in $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$.

Theorem 7.1. *The map Sf_2 on loops in $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ is a homotopy invariant and induces an isomorphism of $\pi_1(\mathcal{F}^1(\mathcal{H}_{\mathbb{R}}))$ with \mathbb{Z}_2 .*

Proof. The argument follows closely [3], p. 11, and [15], as well as Subsection 2.8 of [17]. Let $\rho : \mathcal{F}^1(\mathcal{H}_{\mathbb{R}}) \rightarrow \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ be the (non-linear and discontinuous) map sending T to the partial isometry $W = T|T|^{-1}$ in the polar decomposition. If π denotes the projection onto the Calkin algebra, then the map $\rho_{\mathcal{Q}} = \pi \circ \rho$ sends $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ surjectively onto the space of complex structures in the Calkin algebra given by

$$\mathcal{C}(\mathcal{H}_{\mathbb{R}}) = \{ \pi(J) \in \mathcal{Q}(\mathcal{H}_{\mathbb{R}}) \mid \pi(J)^* = -\pi(J), \pi(J)^*\pi(J) = \mathbf{1} \} .$$

The Bartle-Graves selection theorem (see [5] for a modern proof) provides a right inverse $\theta : \mathcal{C}(\mathcal{H}_{\mathbb{R}}) \rightarrow \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ to $\rho_{\mathcal{Q}}$, namely $\rho_{\mathcal{Q}} \circ \theta = \mathbf{1}$. Moreover, $\theta \circ \rho_{\mathcal{Q}}$ is homotopic to the identity via the homotopy $t \in [0, 1] \mapsto tT + (1-t)\theta(\rho_{\mathcal{Q}}(T)) \in \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$. Thus $\rho_{\mathcal{Q}}$ is actually a homotopy equivalence so that, in particular, $\mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ and $\mathcal{C}(\mathcal{H}_{\mathbb{R}})$ have the same homotopy groups.

Let us next fix a complex structure J on $\mathcal{H}_{\mathbb{R}}$ which also specifies a base point $\rho_{\mathcal{Q}}(J)$ in $\mathcal{C}(\mathcal{H}_{\mathbb{R}})$. Associated to J , one can define a map $\beta_J : \mathcal{O}(\mathcal{H}_{\mathbb{R}}) \rightarrow \mathcal{C}(\mathcal{H}_{\mathbb{R}})$ via $\beta_J(O) = \rho_{\mathcal{Q}}(OJO^*)$. This map is actually a Serre fibration by the argument in Theorem 3.9 of [15]. The fiber over the base point $\rho_{\mathcal{Q}}(J)$ is precisely the set $\mathcal{O}_J(\mathcal{H}_{\mathbb{R}})$ from (5.4). Hence one can dispose of the long exact sequence of homotopy groups, which due to the triviality of the homotopy groups of $\mathcal{O}(\mathcal{H}_{\mathbb{R}})$ implies that the set $\Omega_{\rho_{\mathcal{Q}}(J)}\mathcal{C}(\mathcal{H}_{\mathbb{R}})$ of based loops in the base space is homotopy equivalent to the fiber over the base point which here is $\mathcal{O}_J(\mathcal{H}_{\mathbb{R}})$. Combined with the above, we conclude that

the based loop space $\Omega_J \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ is homotopy equivalent to $\mathcal{O}_J(\mathcal{H}_{\mathbb{R}})$. Since $\pi_0(\mathcal{O}_J(\mathcal{H}_{\mathbb{R}})) = \mathbb{Z}_2$ and due non-trivial examples (Section 9 or Proposition 10.2), this proves the claim. \square

As in [17], one can be more explicit about this map. A loop $t \in [0, 1] \mapsto T_t$ in the skew-adjoint Fredholms based at $T = T_0 = T_1$, say with $\rho_{\mathcal{Q}}(T) = \rho_{\mathcal{Q}}(J)$, pushes down to a based loop in $\mathcal{C}(\mathcal{H}_{\mathbb{R}})$. This lifts to a path in $\mathcal{O}(\mathcal{H}_{\mathbb{R}})$ with endpoints $\mathbf{1}$ and $O \in \mathcal{O}_J(\mathcal{H}_{\mathbb{R}})$. Then $\text{Sf}_2(t \in [0, 1] \mapsto T_t) = j(O)$.

8. AN INDEX FORMULA

The conventional spectral flow can always be expressed as an (Noether) index of an associated Toeplitz operator, see [4, 8, 16, 17]. The following result is the \mathbb{Z}_2 -equivalent of this result.

Theorem 8.1. *Let J be a complex structure and O an orthogonal operator on $\mathcal{H}_{\mathbb{R}}$ such that $[J, O]$ is compact. Extend J to a skew-adjoint operator on $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes \mathbb{C}$ and let P be the spectral projection onto the positive imaginary spectrum. Then*

$$\text{Sf}_2(J, OJO^*) = \dim_{\mathbb{C}} (\text{Ker}_{\mathbb{C}}(POP|_{P\mathcal{H}_{\mathbb{C}}})) \bmod 2 .$$

Here the \mathbb{Z}_2 -index on the r.h.s. is of the type $(j, d) = (2, 8)$ in Theorem 1 of [11] which is proved in Section 2.2.4 therein. Indeed, P satisfies $\bar{P} = \mathbf{1} - P$ (namely, P is even Lagrangian in the terminology of [11]) and $\bar{O} = O$ with complex conjugation in $\mathcal{H}_{\mathbb{C}}$. In particular, the index pairing on the r.h.s. is a homotopy invariant under variations of O and P respecting the two symmetries mentioned above. The proof of Theorem 8.1 is remarkably simple. Both sides of the equality are homotopy invariants and lie in \mathbb{Z}_2 . Hence it is sufficient to verify equality on each component. For $O = \mathbf{1}$, both sides vanish. For the other component, the equality is verified for a non-trivial example in the next section.

9. AN EXAMPLE OF NON-TRIVIAL \mathbb{Z}_2 -VALUED SPECTRAL FLOW

This section introduces the real analogue of classical Toeplitz operators on $L^2(\mathbb{S}^1)$. Furthermore it is an important element of the proof of Theorem 8.1.

Let us consider the real Hilbert space $\mathcal{H}_{\mathbb{R}} = L^2_{\mathbb{R}}(\mathbb{S}^1) \otimes \mathbb{R}^2$, as well as its complexification $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes \mathbb{C} = L^2(\mathbb{S}^1) \otimes \mathbb{C}^2$. The complex conjugation on $\mathcal{H}_{\mathbb{C}}$ is denoted by \mathcal{C} . Next let us consider the discrete Fourier transform

$$\mathcal{F} : \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \rightarrow \mathcal{H}_{\mathbb{C}} , \quad (\mathcal{F}\phi)(k) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{ikn} \phi_n .$$

for $\phi = (\phi_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ with $\phi_n \in \mathbb{C}^2$. Denote the natural complex conjugation on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ also by \mathcal{C} and the reflection on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ by \mathcal{R} (namely $\mathcal{R}\phi = (\phi_{-n})_{n \in \mathbb{Z}}$), one then has $\mathcal{F}\mathcal{C} = \mathcal{R}\mathcal{C}\mathcal{F}$.

Now let us introduce an operator \hat{J} on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ by

$$(9.1) \quad \hat{J} = \imath \text{sgn}(X) \otimes \mathbf{1}_2 + p_0 \otimes \sigma ,$$

where p_n is the projection on the n^{th} component in $\ell^2(\mathbb{Z})$ and

$$\text{sgn}(X) = \sum_{n>0} p_n - \sum_{n<0} p_n$$

is the sign of the position operator $X = \sum_n n p_n$, and $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. All this assures that

$$\hat{J}^2 = -\mathbf{1}, \quad \mathcal{R}\mathcal{C}\hat{J}\mathcal{C}\mathcal{R} = \hat{J}.$$

This implies that $J = \mathcal{F}^* \hat{J} \mathcal{F}$ on $\mathcal{H}_{\mathbb{C}}$ satisfies $J^2 = -\mathbf{1}$ and $\mathcal{C}J\mathcal{C} = J$. Therefore J restricts to $\mathcal{H}_{\mathbb{R}}$ and defines a complex structure there. On basis vectors, it is explicitly given by

$$J \begin{pmatrix} c + \cos(nk) + \sin(mk) \\ c' + \cos(n'k) + \sin(m'k) \end{pmatrix} = \begin{pmatrix} -c' - \text{sgn}(n) \cos(nk) + \text{sgn}(m) \sin(mk) \\ c - \text{sgn}(n') \cos(n'k) + \text{sgn}(m') \sin(m'k) \end{pmatrix},$$

where $c, c' \in \mathbb{R}$ and $n, m, n', m' \in \mathbb{N}$.

Next, another complex structure OJO^* on $\mathcal{H}_{\mathbb{R}}$ will be constructed by conjugating with an orthogonal $O = (O(k))_{k \in \mathbb{S}^1}$ which we choose simply to be

$$O(k) = \begin{pmatrix} \cos(k) & -\sin(k) \\ \sin(k) & \cos(k) \end{pmatrix}.$$

Our first aim in the calculation below is to show that the \mathbb{Z}_2 -valued spectral flow from J to OJO^* along the straight line path $t \in [0, 1] \mapsto J_t = (1-t)J + tOJO^*$ is

$$(9.2) \quad \text{Sf}_2(J, OJO^*) = 1.$$

Actually, it will become apparent that it arises from a single block of the form T_t given in (2.1). It is now also possible to close the path J_t to a closed loop as follows. By Kuipers' theorem there exists a path $t \in [0, 1] \mapsto O_t$ connecting $O_0 = O$ to $O_1 = \mathbf{1}$. Then set $J_t = O_{t-1}JO_{t-1}^*$ for $t \in [1, 2]$. Then $t \in [0, 2] \mapsto J_t$ is a closed loop with non-trivial \mathbb{Z}_2 -valued spectral flow.

The second aim is to show that the associated Toeplitz operator POP has a non-trivial \mathbb{Z}_2 -index:

$$(9.3) \quad \dim_{\mathbb{C}} (\text{Ker}_{\mathbb{C}}(POP|_{P\mathcal{H}_{\mathbb{C}}})) \bmod 2 = 1.$$

Here P is the spectral projection of J viewed as an operator on $\mathcal{H}_{\mathbb{C}}$ corresponding to the eigenvalue \imath . Hence (9.2) and (9.3) together provide an instance in which Theorem 8.1 holds in the non-trivial component.

The verification of (9.2) is easiest on the Fourier transform. Hence let us begin by noting that

$$\hat{O} = \mathcal{F}O\mathcal{F}^* = \frac{1}{2} \begin{pmatrix} S + S^* & \imath(S^* - S) \\ \imath(S - S^*) & S + S^* \end{pmatrix},$$

where S is the right bilateral shift operator on $\ell^2(\mathbb{Z})$. Note that as O is real, one has $\mathcal{R}\mathcal{C}\hat{O}\mathcal{C}\mathcal{R} = \hat{O}$. Furthermore, $S = \mathcal{C}S\mathcal{C} = \mathcal{R}S^*\mathcal{R}$. If now $\pi_n : \mathbb{C} \rightarrow \ell^2(\mathbb{Z})$ denotes the partial isometric

embedding onto the n^{th} component (so that $p_n = \pi_n(\pi_n)^*$), one also has

$$\text{sgn}(X) S = S \text{sgn}(X) + \pi_0(\pi_{-1})^* \otimes \mathbf{1}_2 + \pi_1(\pi_0)^* \otimes \mathbf{1}_2 .$$

Using this and some care and patience, one can check that the first summand in (9.1) satisfies

$$\widehat{O}(\imath \text{sgn}(X) \otimes \mathbf{1}_2) \widehat{O}^* = \imath \text{sgn}(X) \otimes \mathbf{1}_2 + \frac{1}{2}(p_{-1} \otimes (\imath \mathbf{1}_2 - \sigma) - 2p_0 \otimes \sigma + p_1 \otimes (-\imath \mathbf{1}_2 - \sigma)) .$$

Another calculation shows that for the second summand in (9.1)

$$\widehat{O}(p_0 \otimes \sigma) \widehat{O}^* = -\frac{1}{2}(p_{-1} \otimes (\imath \mathbf{1}_2 - \sigma) + p_1 \otimes (-\imath \mathbf{1}_2 - \sigma)) .$$

Note that all these terms are invariant under conjugation with \mathcal{CR} , as they should be. Combining, one deduces

$$\widehat{O} \widehat{J} \widehat{O}^* = \imath \text{sgn}(X) \otimes \mathbf{1}_2 - p_0 \otimes \sigma ,$$

and thus as claimed above

$$\widehat{J}_t = \imath \text{sgn}(X) \otimes \mathbf{1}_2 + (1 - 2t) p_0 \otimes \sigma .$$

The example illustrates that the fundamental spectral unit in this game is a copy of \mathbb{R}^2 . Given a phase W_1 of a skew-adjoint Fredholm, the Hilbert space decomposes as a direct sum of its kernel plus its orthogonal complement. Off the kernel, we can find a basis of the Hilbert space such that the whole Hilbert space is a direct sum of two dimensional subspaces on which W_1 acts as the matrix σ .

Next let us verify (9.3). On the fiber \mathbb{C}^2 acts the Cayley transform $c = 2^{-\frac{1}{2}} \begin{pmatrix} 1 & -\imath \\ & \imath \end{pmatrix}$. Let us set $\widetilde{O} = (\mathbf{1} \otimes c) \widehat{O} (\mathbf{1} \otimes c^*)$ and $\widetilde{P} = (\mathbf{1} \otimes c) \widehat{P} (\mathbf{1} \otimes c^*)$. One can then readily check

$$\widetilde{O} = \begin{pmatrix} S & 0 \\ 0 & S^* \end{pmatrix} , \quad \widetilde{P} = \begin{pmatrix} p_{>} & 0 \\ 0 & p_{\geq} \end{pmatrix} ,$$

where $p_{>} = \sum_{n>0} p_n$ and $p_{\geq} = \sum_{n \geq 0} p_n$. Hence

$$\widetilde{P} \widetilde{O} \widetilde{P} = \begin{pmatrix} p_{>} S p_{>} & 0 \\ 0 & p_{\geq} S^* p_{\geq} \end{pmatrix} .$$

Both entries are unilateral shifts, one left and one right, so that the kernel is indeed of dimension 1, and hence that of POP as well.

10. EXAMPLES OF \mathbb{Z}_2 -SPECTRAL FLOW IN TOPOLOGICAL INSULATORS

Model Hamiltonians for topological insulators are classified by their symmetry type, see *e.g.* [1, 11]. In this paper we consider only two such symmetries, particle-hole symmetry (PHS) and time reversal symmetry (TRS). These are described mathematically by conjugate linear involutions or by conjugate linear complex structures. It is this conjugate linearity that leads to the need for real K -theory and hence \mathbb{Z}_2 -valued spectral flow.

We begin by constructing a Hamiltonian with PHS. Let $T \in \mathcal{F}^1(\mathcal{H}_{\mathbb{R}})$ be a real skew-adjoint Fredholm operator on $\mathcal{H}_{\mathbb{R}}$. Its extension as a real linear operator to the complexified Hilbert

space $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes \mathbb{C}$ is still denoted by T . Associated to this T , one can define a Hamiltonian of Bogoliubov-de Gennes (BdG) type [1] in the so-called Majorana representation by

$$(10.1) \quad H_{\text{Maj}} = \imath T .$$

This is a self-adjoint operator acting on $\mathcal{H}_{\mathbb{C}}$. The corresponding Atiyah-Singer \mathbb{Z}_2 -index (3.1) is

$$\text{Ind}_1(H_{\text{Maj}}) = \dim_{\mathbb{C}}(\text{Ker}_{\mathbb{C}}(H_{\text{Maj}})) \bmod 2 .$$

This separates the set of essentially gapped BdG Hamiltonians into two sets, those with an even number of zero (Majorana) modes, and those with an odd number. Now let us furnish $\mathcal{H}_{\mathbb{C}}$ with some non-trivial grading and let C be the Cayley transformation in that grading

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\imath \\ 1 & \imath \end{pmatrix} .$$

This then brings H_{Maj} into the more conventional BdG form:

$$H = C H_{\text{Maj}} C^* .$$

Recall that for any operator A , we set $\bar{A} = C A C$ and $A^t = \bar{A}^*$. The BdG equation is then

$$(10.2) \quad K^* \bar{H} K = -H , \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Now the upper component is interpreted as particles, and the lower one as anti-particles. Given a one-parameter family $t \in [0, 1] \mapsto H_t$ of BdG-operators with $0 \notin \sigma_{\text{ess}}(H_t)$ for all $t \in [0, 1]$ and $0 \notin \sigma(H_0) \cup \sigma(H_1)$, one can consider the associated \mathbb{Z}_2 -valued spectral flow. In the following, we will construct two examples of non-trivial \mathbb{Z}_2 -valued spectral flow in one-dimensional BdG Hamiltonians. One is linked to a flux tube argument allowing us to describe zero modes attached to a defect in the model, the other to a cycle used for orbital polarization.

10.1. Flux tube through a Kitaev chain. The infinite and ‘clean’ (meaning no disorder) Kitaev chain is described by a Hamiltonian on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ given by

$$(10.3) \quad H = \frac{1}{2} \begin{pmatrix} S + S^* + 2\mu & \imath(S - S^*) \\ \imath(S - S^*) & -(S + S^* + 2\mu) \end{pmatrix} .$$

Here S denotes the right shift and $\mu \in \mathbb{R}$ is a chemical potential. Using the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

one has

$$(10.4) \quad H = S_0 + S_0^* + \mu \mathbf{1} \otimes \sigma_3 , \quad S_0 = S \otimes \frac{1}{2}(\sigma_3 + \imath \sigma_1) = S \otimes \frac{1}{2} \begin{pmatrix} 1 & \imath \\ \imath & -1 \end{pmatrix} .$$

The operator S_0 is the right shift on the line with particle-hole fiber, with coupling terms going from the particle to the hole fiber and visa versa. The index 0 indicates that there is no flux pushed through. Below, the definition will be extended to S_α with a flux $2\pi\alpha$. For $\mu \neq \{-1, 1\}$ one has $0 \notin \sigma(H)$.

The Hamiltonian H has an even PHS (that is, the symmetry is given by an involution and thus its square is the identity) with $K = \sigma_1$ as in (10.2) and an even TRS (again the operator giving the TRS symmetry squares to the identity) with $J = \sigma_3$. Consequently, in the usual symmetry classification scheme [1], this model lies in the Class BDI. As it also has an induced ‘chiral symmetry’ given by the composition of the PHS and TRS, it has a well-defined \mathbb{Z} -index [18]. This reduces to a \mathbb{Z}_2 index if the TRS is broken and the system then only lies in the Class D [11]. This \mathbb{Z}_2 index is equal to the number of zero modes (Majorana states) that a half-space restriction of the model has, taken modulo 2. All these facts are stable under perturbations by disorder, as long as the stated symmetries hold. For sake of simplicity, we will first work with the clean model and then add perturbations by homotopy at the end of the section.

Now let us consider the model as defined on a (discrete) strip $\mathbb{Z} \times \{+, -\}$ of width 2. A magnetic flux $2\pi\alpha \in [0, 2\pi)$ will be inserted in the cell between $0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$. This can be realised by various gauge potentials, namely a real-valued function on the oriented links of the underlying lattice $\mathbb{Z} \times \{+, -\}$. It is given by a function $A : (\mathbb{Z} \times \{+, -\})^2 \rightarrow \mathbb{R}$ which is anti-symmetric in its two arguments and is only non-vanishing for two neighbouring sites of the lattice $\mathbb{Z} \times \{+, -\}$ (see Section 2.1 of [9] for a concise review of the basic facts on gauges and gauge transformations used below). We will actually use two different gauges which are represented graphically in Fig. 2. The first gauge is

$$A((n, \eta), (n', \eta')) = \pi\alpha \delta_{n,1} \delta_{n',0} \delta_{\eta,-} \delta_{\eta',-} - \pi\alpha \delta_{n,1} \delta_{n',0} \delta_{\eta,+} \delta_{\eta',+},$$

where $(n, \eta), (n', \eta') \in \mathbb{Z} \times \{+, -\}$. Following again [9], the Hamiltonian with a flux tube is in this gauge given by

$$(10.5) \quad H_\alpha = S_\alpha + S_\alpha^* + \mu \mathbf{1} \otimes \sigma_3, \quad S_\alpha = S_0 + \pi_1(\pi_0)^* \otimes \frac{1}{2} \begin{pmatrix} e^{-i\pi\alpha} - 1 & i(e^{i\pi\alpha} - 1) \\ i(e^{-i\pi\alpha} - 1) & -(e^{i\pi\alpha} - 1) \end{pmatrix},$$

where the partial isometry π_n onto site n are defined as in Section 9. The second (non-local) gauge is

$$\tilde{A}((n, \eta), (n', \eta')) = \pi\alpha \delta_{n,n'} \delta_{n \leq 0} \delta_{\eta,-} \delta_{\eta',+} - \pi\alpha \delta_{n,n'} \delta_{n \geq 1} \delta_{\eta,-} \delta_{\eta',+},$$

and the Hamiltonian

$$(10.6) \quad \tilde{H}_\alpha = \tilde{S}_\alpha + \tilde{S}_\alpha^* + \mu \mathbf{1} \otimes \sigma_3, \quad \tilde{S}_\alpha = \frac{1}{2} \begin{pmatrix} \mathbf{1} & i e^{i\pi\alpha \operatorname{sgn}'(X)} \\ i e^{-i\pi\alpha \operatorname{sgn}'(X)} & -\mathbf{1} \end{pmatrix} \cdot S \otimes \mathbf{1}_2,$$

where $\operatorname{sgn}'(X) = \operatorname{sgn}(X) + p_0$ is the modified sign of the position operator with $\operatorname{sgn}'(X)\pi_0 = \pi_0$ (cf. Section 9). First of all, let us note that both $\alpha \mapsto H_\alpha$ and $\alpha \mapsto \tilde{H}_\alpha$ are paths in Hamiltonians with the even BdG symmetry (10.2), and thus after Cayley transform paths in $\mathcal{F}^1(\mathcal{H}_\mathbb{R})$. Second

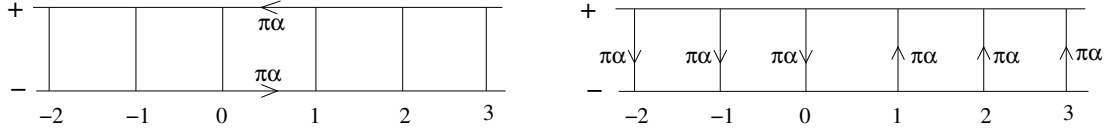


FIGURE 2. The two gauges A and \tilde{A} used for a flux $2\pi\alpha$ through the cell between 0 and 1.

of all, as A and \tilde{A} induce the same magnetic field, there exists a so-called gauge transformation $G : \mathbb{Z} \times \{-, +\} \rightarrow \mathbb{R}$ such that, for $|(n, \eta) - (n', \eta')| = 1$,

$$(10.7) \quad \tilde{A}((n, \eta), (n', \eta')) = A((n, \eta), (n', \eta')) + G(n, \eta) - G(n', \eta') ,$$

and via this gauge transformation the Hamiltonians are unitarily equivalent:

$$(10.8) \quad \tilde{H}_\alpha = e^{iG(X, \eta)} H_\alpha e^{-iG(X, \eta)} .$$

Let us collect a few facts related to these Hamiltonians.

Proposition 10.1. *The following holds for all $\alpha \in \mathbb{R}$ with $K = \mathbf{1} \otimes \sigma_1$ and $I = \mathbf{1} \otimes \sigma_3$.*

- (i) $K^* \mathcal{C} H_\alpha \mathcal{C} K = -H_\alpha$ and $\sigma(H_\alpha) = -\sigma(H_\alpha)$
- (ii) $K^* \mathcal{C} \tilde{H}_\alpha \mathcal{C} K = -\tilde{H}_\alpha$ and $\sigma(\tilde{H}_\alpha) = -\sigma(\tilde{H}_\alpha)$
- (iii) $H_\alpha - H_0$ is compact and $\sigma_{\text{ess}}(H_\alpha) = \sigma_{\text{ess}}(H_0)$
- (iv) $\sigma(H_\alpha) = \sigma(\tilde{H}_\alpha)$
- (v) $H_{\alpha+2} = H_\alpha$
- (vi) $\tilde{H}_1 = I^* \tilde{H}_0 I$ and $\sigma(\tilde{H}_1) = \sigma(\tilde{H}_0)$
- (vii) $\mathcal{C} \tilde{H}_\alpha \mathcal{C} = \tilde{H}_{1-\alpha}$ and $\sigma(H_\alpha) = \sigma(H_{1-\alpha})$
- (viii) $H_1 + H_0 = \hat{H}_l \oplus \hat{H}_r$ where \hat{H}_l and \hat{H}_r are the restrictions of H_0 to the left and right half-line Hilbert spaces $\ell^2(\mathbb{N}_{\leq 0}) \otimes \mathbb{C}^2$ and $\ell^2(\mathbb{N}_{\geq 1}) \otimes \mathbb{C}^2$ respectively, both with Dirichlet boundary conditions.
- (ix) For $|\mu| < 1$, $\dim_{\mathbb{C}}(\text{Ker}_{\mathbb{C}}(\hat{H}_l)) = \dim_{\mathbb{C}}(\text{Ker}_{\mathbb{C}}(\hat{H}_r)) = 1$ and $\dim_{\mathbb{C}}(\text{Ker}_{\mathbb{C}}(H_0 + H_1)) = 2$

Proof. Items (i) and (ii) can be checked algebraically from (10.5) and (10.6), and (iii) follows from (10.5) combined with Weyl's theorem of the essential spectrum. Furthermore, (iv) results from the gauge transformation (10.8) and (v) is obvious. Items (vi) and (vii) are due to the identity

$$e^{i\pi \text{sgn}'(X)} = e^{-i\pi \text{sgn}'(X)} = -\mathbf{1} .$$

For (viii), one uses

$$S_1 = S_0 + \pi_1(\pi_0)^* \otimes \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix} .$$

Comparing with (10.4), one sees that the second summand reverses the links between $0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$, so that $S_0 + S_1$ is indeed a direct sum of two uni-lateral shifts (tensorized with a 2×2 matrix). Finally (ix) results from the fact that both \hat{H}_l and \hat{H}_r have exactly one Majorana boundary state for $|\mu| < 1$, see Section 3.6.1 of [11]. \square

Now we can consider the \mathbb{Z}_2 -valued spectral flow along the path $\alpha \in [0, 1] \mapsto H_\alpha$, strictly speaking to the path of real skew-adjoints $\alpha \in [0, 1] \mapsto T_\alpha = -\imath C^* H_\alpha C$ of the associated Majorana representation (10.1).

Proposition 10.2. *For $|\mu| < 1$,*

$$\text{Sf}_2(\alpha \in [0, 1] \mapsto H_\alpha) = 1.$$

Proof. Let us first argue that

$$\text{Sf}_2(\alpha \in [0, 1] \mapsto H_\alpha) = \text{Sf}_2(t \in [0, 1] \mapsto H(t) = (1-t)H_0 + tH_1).$$

As $\alpha \in [0, 1] \mapsto H_\alpha$ and the straight line path $H(t) = (1-t)H_0 + tH_1$ have the same end points, this follows from the homotopy invariance (Theorem 4.3) once we have provided a homotopy between the two paths. But $H_\alpha = H_0 + L_\alpha$ for some finite range L_α , see (10.5), and on the other hand $H(t) = H_0 + L'_t$ for another finite range L'_t . Hence a homotopy of paths with fixed end points is given by $s \in [0, 1] \mapsto H(t, s) = H_0 + sL_t + (1-s)L'_t$. In conclusion, it remains to show $\text{Sf}_2(t \in [0, 1] \mapsto H(t)) = 1$. For all $|\mu| < 1$, the zero energy 0 lies in a gap of H_0 and H_1 . Hence there is no \mathbb{Z}_2 -spectral flow when μ is homotopically deformed to $\mu = 0$, so that it is sufficient to consider this case. But for $\mu = 0$, one can check starting from (10.3) that $(H_0)^2 = \mathbf{1}$. By Proposition 10.1 and the gauge transformation (10.8) one also concludes $(H_1)^2 = \mathbf{1}$. Therefore the real skew-adjoints associated to the Majorana representation (10.1) are actually complex structures, so that one can invoke Proposition 6.2 to calculate $\text{Sf}_2(t \in [0, 1] \mapsto H(t)) = \text{Sf}_2(H_0, H_1)$ using Proposition 10.1(ix). \square

The main application will concern the spectral properties at $\alpha = \frac{1}{2}$. This is particularly interesting because the Hamiltonian at $\alpha = \frac{1}{2}$ has the even TRS $\mathcal{C}\tilde{H}_{\frac{1}{2}}\mathcal{C} = \tilde{H}_{\frac{1}{2}}$. It thus lies in the same universality class BDI as the Kitaev model, and is, in fact, obtained from the latter model by the perturbation of a local defect at the sites $0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$. The following result shows that this defect necessarily has zero modes attached to it. As $\text{Ind}_1(H_\alpha) = 0$ for all α , these zero modes are always evenly degenerate.

Theorem 10.3. *For $|\mu| < 1$, the Hamiltonian $H_{\frac{1}{2}}$ has an odd number of evenly degenerate zero eigenvalues:*

$$\frac{1}{2} \dim_{\mathbb{C}}(\text{Ker}_{\mathbb{C}}(H_{\frac{1}{2}})) \bmod 2 = 1.$$

Proof. The symmetry property $\mathcal{C}\tilde{H}_\alpha\mathcal{C} = \tilde{H}_{1-\alpha}$ Proposition 10.1(vii) combined with the last claim of Theorem 4.2 shows that for any $\alpha_0 < \frac{1}{2}$ such that H_{α_0} has minimal kernel dimension,

$$\text{Sf}_2(\alpha \in [0, \alpha_0] \mapsto H_\alpha) = \text{Sf}_2(\alpha \in [1 - \alpha_0, 1] \mapsto H_\alpha).$$

Hence from the concatenation property

$$\text{Sf}_2(\alpha \in [0, 1] \mapsto H_\alpha) = \text{Sf}_2(\alpha \in [\alpha_0, 1 - \alpha_0] \mapsto H_\alpha).$$

As H_α is analytic in α , one can conclude that at $\alpha = \frac{1}{2}$ there is an odd number of eigenvalue crossings by Proposition 10.2. \square

Finally, let us comment on adding disorder or other perturbations to the Hamiltonian:

$$H_\alpha(\lambda) = H_\alpha + \lambda V .$$

Here V is a possibly random perturbation representing disorder in the system but having the BdG and TRS symmetries $K^* \bar{V} K = -V$ and $I^* \bar{V} I = V$. Furthermore, λ is chosen to be sufficiently small such that the gap in the essential spectrum of H_0 does not close. This essentially imposes that $|\lambda| \leq C |\mu|$ for some suitable constant depending on the norm of V . Now all statements proved above directly transpose to $H_\alpha(\lambda)$, in particular, also Theorem 10.3.

10.2. \mathbb{Z}_2 -polarization. This short last section indicates that there are further natural instances where the \mathbb{Z}_2 -valued spectral flow appears. It is based on several prior results which are only briefly described and the reader is urged to read up in the cited references. Let us begin by recalling from [18] the connection between polarization and spectral flow of half-sided operators based on the bulk-boundary correspondence. We restrict to dimension 1, even though this connection holds in any dimension. Suppose given a smooth loop $t \in [0, 2\pi) \mapsto h_t$ of periodic Hamiltonians on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$. The orbital polarization at Fermi level μ lying in a gap of h_t for all t is defined by

$$\Delta P = \imath \int_0^{2\pi} dt \operatorname{Tr}_N (\pi_0^* p_t [\partial_t p_t, \imath [X, p_t]] \pi_0) ,$$

where $p_t = \chi(h_t \leq \mu)$ is the (instantaneous) Fermi projection. Actually, the polarization ΔP is defined as the accumulated charge during one loop and then the above is the so-called King-Smith-Vanderbilt formula which holds in the adiabatic limit as proved in [20].

The second important fact for the following is that ΔP is 2π times an integer called the Chern number $\operatorname{Ch}(p)$ where $p = (p_t)_{t \in [0, 2\pi)}$. Again this Chern number is also given as the index of a certain Fredholm operator [18]. Next let us consider the restrictions $\widehat{h}(t)$ of $h(t)$ to $\ell^2(\mathbb{N}) \otimes \mathbb{C}^N$, say, with Dirichlet boundary conditions (actually any local boundary condition will do). These operators may have bound states and it is proved in [18] that the standard complex spectral flow of these eigenvalues through μ is equal to the Chern number, namely

$$\Delta P = 2\pi \operatorname{Sf}(t \in [0, 2\pi) \mapsto \widehat{h}_t \text{ by } \mu) .$$

Based on these facts, we now construct an example of a BdG-Hamiltonian with non-trivial \mathbb{Z}_2 -valued spectral flow. On $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2N}$, let us set

$$H_t(\lambda) = \begin{pmatrix} h_t & 0 \\ 0 & -\bar{h}_t \end{pmatrix} + \lambda V_t ,$$

where $\lambda \geq 0$ is a coupling constant and $t \in [0, 2\pi) \mapsto V_t$ is a smooth loop of bounded BdG operators, namely $K^* \bar{V}_t K = -V_t$. Then also $H_t(\lambda)$ satisfies the BdG equation (10.2). In the following, it will be assumed that λ is sufficiently small such that $0 \notin \sigma(H_t(\lambda))$ for all $t \in [0, 2\pi)$. Due to the minus sign, the polarization $\Delta P(\lambda)$ of $t \in [0, 2\pi) \mapsto H_t(\lambda)$ vanishes and is equal to

the spectral flow of the half-line restriction $\widehat{H}_t(\lambda)$ to $\ell^2(\mathbb{N}) \otimes \mathbb{C}^{2N}$:

$$\Delta P(\lambda) = 2\pi \text{Sf}(t \in [0, 2\pi) \mapsto \widehat{H}_t(\lambda)) = 0.$$

Note that the vanishing of the spectral flow on the r.h.s. also follows immediately from the BdG symmetry which implies that the spectrum of $\widehat{H}_t(\lambda)$ is always symmetric around 0 so that the spectral flow necessarily vanishes. For non-trivial V_t and say intermediate $\lambda \neq 0$ (still such that the gap remains open) it may not be possible to define the spectral flow of the upper left and lower left components separately any more. Nevertheless the \mathbb{Z}_2 -valued spectral flow may be topologically non-trivial, as shows the following result.

Theorem 10.4. *For λ sufficiently small, the \mathbb{Z}_2 -valued spectral flow satisfies*

$$\text{Sf}_2(t \in [0, 2\pi) \mapsto \widehat{H}_t(\lambda)) = \frac{1}{2\pi} \Delta P \bmod 2 \in \mathbb{Z}_2.$$

This quantity is called the \mathbb{Z}_2 -polarization. In particular, if it is equal to 1, there is at least one $t \in (0, 2\pi)$ such that \widehat{H}_t has a zero mode with multiplicity equal to 2 modulo 4.

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